

Gelfand Models for Diagram Algebras

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Abstract

A Gelfand model for a semisimple algebra A over \mathbb{C} is a complex linear representation that contains each irreducible representation of A with multiplicity exactly one. We give a method of constructing these models that works uniformly for a large class of semisimple, combinatorial diagram algebras including: the partition, Brauer, rook monoid, rook-Brauer, Temperley-Lieb, Motzkin, and planar rook monoid algebras. In each case, the model representation is given by diagrams acting via “signed conjugation” on the linear span of their vertically symmetric diagrams. This representation is a generalization of the Saxl model for the symmetric group, and, in fact, our method is to use the Jones basic construction to lift the Saxl model from the symmetric group to each diagram algebra. In the case of the planar diagram algebras, our construction exactly produces the irreducible representations of the algebra.

Keywords: Gelfand model; multiplicity-free representation; symmetric group; partition algebra; Brauer algebra; Temperley-Lieb algebra; Motzkin algebra; rook-monoid.

Introduction

A famous consequence of Robinson-Schensted-Knuth (RSK) insertion is that the set of standard Young tableaux with k boxes is in bijection with the set of involutions in the symmetric group S_k (the permutations $\sigma \in S_k$ with $\sigma^2 = 1$). Furthermore, these standard Young tableaux index the bases for the irreducible $\mathbb{C}S_k$ modules, so it follows that the sum of the degrees (dimensions) of the irreducible S_k modules equals the number of involutions in S_k . This suggests the possibility of a representation of the symmetric group on the linear span of its involutions which decomposes into irreducible S_k -modules such that the multiplicity of each irreducible is exactly 1. Indeed, Saxl [Sxl] and Kljačko [Klj] have constructed such a module under which the symmetric group acts on its involutions by a twisted, or signed, conjugation (see Section 2.2). A combinatorial construction of an analogous module was studied recently by Adin, Postnikov, and Roichman [APR] and extended to the rook monoid and related semigroups in [KM]. A representation for which each irreducible appears with multiplicity one is called a *Gelfand model* (or, simply, a *model*), because of the work in [BGG] on models for complex Lie groups.

In [HL] the RSK algorithm is extended to work for a large class of well-known, combinatorial diagram algebras including the partition, Brauer, rook monoid, rook-Brauer, Temperley-Lieb,

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Motzkin, and planar rook monoid algebras. A consequence [HL, (5.5)] of this algorithm is that the sum of the degrees of the irreducible representations of each of these algebras equals the number of horizontally symmetric basis diagrams in the algebra. This suggests the existence of a model representation of each of these algebras on the span of its symmetric diagrams, and the main result of this paper is to produce a such a model.

Let A_k denote one of the following unital, associative \mathbb{C} -algebras: the partition, Brauer, rook monoid, rook-Brauer, Temperley-Lieb, Motzkin, or planar rook monoid algebra. Then A_k has a basis of diagrams and a multiplication given by diagram concatenation. The algebra A_k depends on a parameter $x \in \mathbb{C}$ and is semisimple for all but a finite number of choices of x . When A_k is semisimple, its irreducible modules are indexed by a set Λ_{A_k} , and for $\lambda \in \Lambda_{A_k}$, we let A_k^λ denote the irreducible A_k -module labeled by λ . We construct, in a uniform way, an A_k -module M_{A_k} which decomposes as $M_{A_k} \cong \bigoplus_{\lambda \in \Lambda_{A_k}} A_k^\lambda$.

Our model representation is constructed as follows. For a basis diagram d , let d^T be its reflection across its horizontal axis and say that a diagram t is symmetric if $t^T = t$. A basis diagram d acts on a symmetric diagram t by “signed conjugation”: $d \cdot t = S(d, t) d t d^T$, where $S(d, t)$ is the sign on the permutation of the fixed blocks of t induced by conjugation by d (see Example 3.19 for details). In each example, our basis diagrams are assigned a rank, which is the number of blocks in the diagram that propagate from the top row to the bottom row. We let $M_{A_k}^r$ be the linear span of the symmetric diagrams of rank r and our model is the direct sum $M_{A_k} = \bigoplus_{r=0}^k M_{A_k}^r$.

The diagram algebras in this paper naturally form a tower $A_0 \subseteq A_1 \subseteq \dots \subseteq A_k$, and we are able to use the structure of the Jones basic construction of this tower to derive our model. Each algebra contains a basic construction ideal $J_{k-1} \subseteq A_k$ such that $A_k \cong J_{k-1} \oplus C_k$, where $C_k \cong \mathbb{C}S_k$ for nonplanar diagram algebras and $C_k \cong \mathbb{C}\mathbf{1}_k$ for planar diagram algebras. The ideal J_{k-1} is in Schur-Weyl duality with one of A_{k-1} or A_{k-2} (depending on the specific diagram algebra). In this setup, we are able to take models for each C_r , $0 \leq r \leq k$, and lift them to a model for A_k .

For the planar diagram algebras — the Temperley-Lieb, Motzkin, and planar rook monoid algebras — the algebra $C \cong \mathbb{C}\mathbf{1}_k$ is trivial and the model is trivial. It follows that $M_{A_k}^r$ is irreducible and that signed conjugation produces a complete set of irreducible modules for the planar algebras. For the nonplanar diagram algebras, the algebra is $C \cong \mathbb{C}S_k$, and we use the Saxl model for S_r . In this case $M_{A_k}^r$ is further graded as $M_{A_k}^r = \bigoplus_f M_{A_k}^{r,f}$, where $M_{A_k}^{r,f}$ is the linear span of symmetric diagrams of rank r having f “fixed blocks,” and $M_{A_k}^{r,f}$ decomposes into irreducibles labeled by partitions having f odd parts.

Besides being natural constructions, these representations are useful in several ways: (1) In a model representation, isotypic components are irreducible components, so projection operators map directly onto irreducible modules without being mixed up among multiple isomorphic copies of the same module. (2) A key feature of our model is that we give the explicit action of each basis element of A_k on the basis of $M_{A_k}^{r,f}$. For small values of k , and for all values of k in the planar case, these representations are irreducible or have few irreducible components. Thus, in practice, the model provides a natural and easy way to compute the explicit action of basis diagrams on irreducible representations. Indeed, it is through this construction that the irreducible modules for the Motzkin [BH], the rook-Brauer [dH], and the planar rook monoid [FHH] were discovered. (3) Gelfand models are useful in the study of Markov chains on related combinatorial objects; see, for example, Chapter 3F of [Di] and the references therein, as well as [DH], [RSW].

Finally, the enumeration of symmetric diagrams in these algebras according to rank and number of fixed blocks gives rise to well-known, interesting integer sequences. These combinatorics are analyzed in Section 4, where we work out the details of the model representation for each algebra.

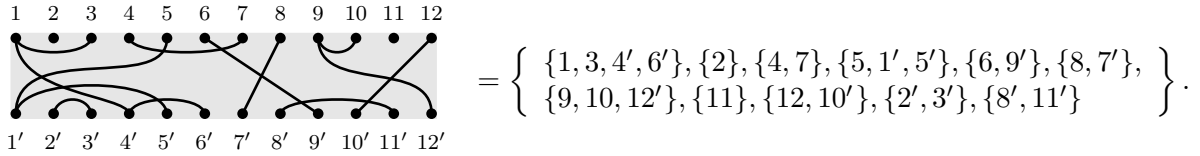
Acknowledgements. We thank Arun Ram for suggesting that we look for model representations of these algebras after seeing the dimension results in [HL]. We also thank Michael Decker [Dec], whose honors project, under the direction of the first author of this paper, examined the model characters of the symmetric group and the partition algebra. It was during this collaboration that we constructed the combinatorial Saxl model for the symmetric group and conjectured the general construction of Gelfand models for diagram algebras.

1 The Partition Algebra and its Diagram Subalgebras

In this section, we describe the partition monoid \mathcal{P}_k and the Martin-Jones partition algebra $\mathcal{P}_k(x)$ over \mathbb{C} with a parameter $x \in \mathbb{C}$. The other diagram algebras of interest in this paper — the Brauer, rook-monoid, symmetric group, rook-Brauer, Temperley-Lieb, Motzkin, and planar rook monoid algebras — are all realized as diagram subalgebras of $\mathcal{P}_k(x)$.

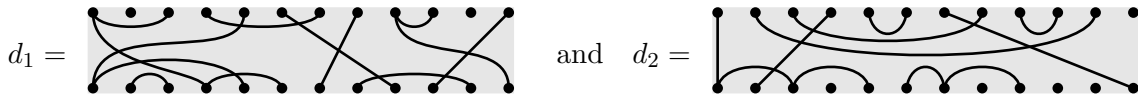
1.1 The partition monoid \mathcal{P}_k

For $k \in \mathbb{Z}_{>0}$, let \mathcal{P}_k denote the set of set partitions of $\{1, 2, \dots, k, 1', 2', \dots, k'\}$. We represent a set partition $d \in \mathcal{P}_k$ by a diagram with vertices in the top row labeled $1, \dots, k$ and vertices in the bottom row labeled $1', \dots, k'$. Assign edges in this diagram so that the connected components equal the underlying set partition d . For example, the following is a diagram $d \in \mathcal{P}_{12}$,



We refer to the parts of a set partition as *blocks*, so that the above diagram has 11 blocks. The diagram of d is not unique, since it only depends on the underlying connected components. We make the following convention: if a block contains vertices from both the top row and bottom row, then *we always connect the leftmost vertex in the top row of a block with the leftmost vertex in the bottom row of the block by a single vertical edge*.

We multiply two set partition diagrams $d_1, d_2 \in \mathcal{P}_k$ as follows. Place d_1 above d_2 and identify each vertex j' in the bottom row of d_1 with the corresponding vertex j in the top row of d_2 . Remove any connected components that live entirely in the middle row and let $d_1 \circ d_2 \in \mathcal{P}_k$ be the resulting diagram. For example, if



then

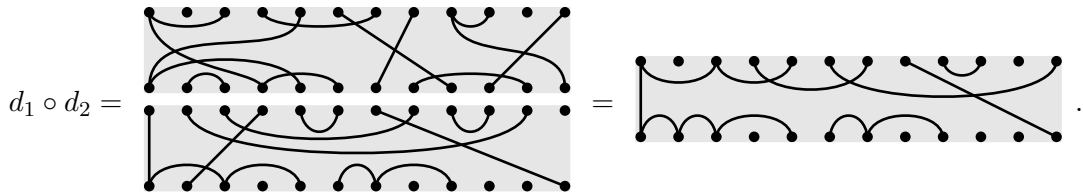


Diagram multiplication is associative and makes \mathcal{P}_k a monoid with identity $1_k = \text{[Diagram 1_k]}$.

1.2 The partition algebra $P_k(x)$

Now let $x \in \mathbb{C}$, define $P_0(x) = \mathbb{C}$, and for $k \geq 1$, let $P_k(x)$ be the \mathbb{C} -vector space with basis \mathcal{P}_k . If $d_1, d_2 \in \mathcal{P}_k$, let $\kappa(d_1, d_2)$ denote the number of connected components that are removed from the middle row in computing $d_1 \circ d_2$, and define

$$d_1 d_2 = x^{\kappa(d_1, d_2)} d_1 \circ d_2. \quad (1.1)$$

In the multiplication example of the previous section $\kappa(d_1, d_2) = 1$ and $d_1 d_2 = x(d_1 \circ d_2)$. This product makes $P_k(x)$ an associative algebra with identity $\mathbf{1}_k$.

We say that a block B in a set partition diagram $d \in \mathcal{P}_k$ is a *propagating* block if B contains vertices from both the top and bottom row of d ; that is, both $B \cap \{1, 2, \dots, k\}$ and $B \cap \{1', 2', \dots, k'\}$ are nonempty. The *rank* of $d \in \mathcal{P}_k$ (also called the *propagating number*) is

$$\text{rank}(d) = (\text{the number of propagating blocks in } d). \quad (1.2)$$

The rank satisfies

$$\text{rank}(d_1 d_2) \leq \min(\text{rank}(d_1), \text{rank}(d_2)). \quad (1.3)$$

For $0 \leq r \leq k$, we let $J_r \subseteq P_k(x)$ be the \mathbb{C} -span of the diagrams of rank *less than or equal to* r . Then J_r is a two-sided ideal in $P_k(x)$, and we have a tower of ideals: $J_0 \subseteq J_1 \subseteq J_2 \subseteq \dots \subseteq J_k = P_k(x)$.

The partition algebra was first defined independently by P.P. Martin [Ma] and V.F.R. Jones [Jo2] as a higher-dimension generalization of the Temperley-Lieb algebra in statistical mechanics. The partition algebra $P_k(n)$ is in Schur-Weyl duality with the symmetric group S_k on tensor space (see also [HR2] for a survey of many results on the partition algebra).

1.3 Subalgebras

For each $k \in \mathbb{Z}_{>0}$, the following are subalgebras of the partition algebra $P_k(x)$:

$$\begin{aligned} \mathbb{C}S_k &= \mathbb{C}\text{-span}\{d \in \mathcal{P}_k \mid \text{rank}(d) = k\}, \\ B_k(x) &= \mathbb{C}\text{-span}\{d \in \mathcal{P}_k \mid \text{all blocks of } d \text{ have size } 2\}, \\ R_k &= \mathbb{C}\text{-span}\left\{d \in \mathcal{P}_k \left| \begin{array}{l} \text{all blocks of } d \text{ have at most one vertex in } \{1, \dots, k\} \\ \text{and at most one vertex in } \{1', \dots, k'\} \end{array} \right. \right\}, \\ RB_k(x) &= \mathbb{C}\text{-span}\{d \in \mathcal{P}_k \mid \text{all blocks of } d \text{ have size } 1 \text{ or } 2\}. \end{aligned}$$

Here, $\mathbb{C}S_k$ is the group algebra of the symmetric group, $B_k(x)$ is the Brauer algebra [Br], R_k is the rook monoid algebra [So], and $RB_k(x)$ is the rook-Brauer algebra [dH], [MM].

A set partition is *planar* if it can be represented as a diagram without edge crossings inside of the rectangle formed by its vertices. The planar partition algebra [Jo2] is

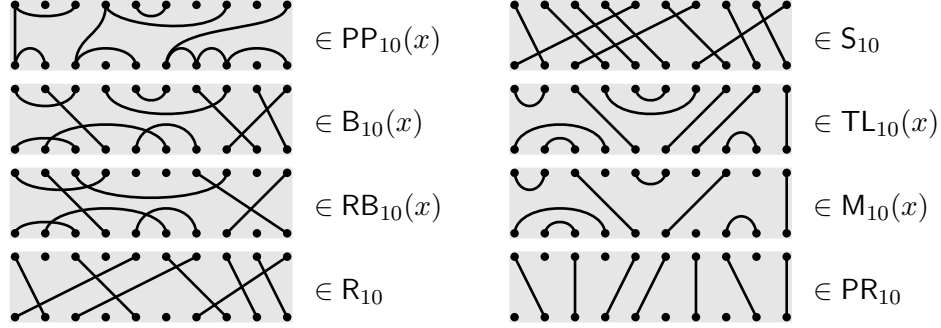
$$PP_k(x) = \mathbb{C}\text{-span}\{d \in \mathcal{P}_k \mid d \text{ is planar}\}.$$

The following are the planar subalgebras of $P_k(x)$:

$$\begin{aligned} \mathbb{C}\{\mathbf{1}_k\} &= \mathbb{C}S_k \cap PP_k(x), & \text{TL}_k(x) &= B_k(x) \cap PP_k(x), \\ \text{PR}_k &= R_k \cap PP_k(x), & \text{M}_k(x) &= RB_k(x) \cap PP_k(x). \end{aligned}$$

Here, $\text{TL}_k(x)$ is the Temperley-Lieb algebra [TL], PR_k is the planar rook monoid algebra [FHH], and $\text{M}_k(x)$ is the Motzkin algebra [BH]. The parameter x does not arise when multiplying symmetric group diagrams (as there are never middle blocks to be removed). The parameter is set to be $x = 1$

for the rook monoid algebra and the planar rook monoid algebra. Here are examples from each of these subalgebras:



2 A Model Representation of the Symmetric Group

2.1 Saxl's model characters of S_k

An involution $t \in S_k$ is a permutation such that $t^2 = 1$. In disjoint cycle notation, involutions consist of 2-cycles and fixed points. Let I_k be the set of involutions in S_k and let I_k^f be the involutions in S_k which fix precisely f points; that is,

$$I_{S_k} = \{ t \in S_n \mid t^2 = 1 \}, \quad (2.1)$$

$$I_{S_k}^f = \{ t \in S_n \mid t^2 = 1 \text{ and } t \text{ has } f \text{ fixed points} \}. \quad (2.2)$$

For a fixed involution $t \in I_k^f$, let $C(t) \subseteq S_n$ be the centralizer of t in S_k . Consider the group action of S_k on itself by conjugation so that $w \cdot \sigma = w\sigma w^{-1}$ for all $w, \sigma \in S_k$. Then, under this action, $C(t)$ is the stabilizer of t , and $I_{S_k}^f$ is the orbit of t , so $|S_k| = |C(t)| \cdot |I_{S_k}^f|$. The number of involutions in S_k having $f = k - 2\ell$ fixed points and ℓ transpositions is counted by

$$|I_{S_k}^f| = |I_{S_k}^{k-2\ell}| = \binom{k}{2\ell} (2\ell - 1)!!, \quad (2.3)$$

where $(2\ell - 1)!! = (2\ell - 1)(2\ell - 3) \cdots 3 \cdot 1$.

If $w \in C(t)$, then $wtw^{-1} = t$, so w fixes t but possibly permutes the fixed points of t . Let π_f be the linear character of $C(t)$ such that $\pi_f(w)$ is the sign of the permutation of w on the fixed points of t . Saxl [Sxl] (see also [Klj] or [IRS]) proves the following decomposition of the induced character

$$\varphi_{S_k}^f := \text{Ind}_{C(t)}^{S_n}(\pi_f) = \sum_{\substack{\lambda \vdash k \\ \text{odd}(\lambda)=f}} \chi_{S_k}^\lambda, \quad (2.4)$$

where $\text{odd}(\lambda)$ is the number of odd parts of the partition λ . Thus,

$$\varphi_{S_k} := \sum_{\ell=0}^{\lfloor k/2 \rfloor} \varphi_{S_k}^{k-2\ell} = \sum_{\lambda \vdash k} \chi_{S_k}^\lambda. \quad (2.5)$$

is a model character for S_k . This result generalizes the classic result (see [Th, Theorem IV] or [JK, 5.4.23]) for fixed-point-free permutations, i.e., the case where $f = 0$. In this case, there are no fixed points and π_0 is the trivial character of $C(t)$.

If we let $s_k = |S_k| = \sum_{\ell=0}^{\lfloor k/2 \rfloor} |I_{S_k}^{k-2\ell}|$ denote the total number of involutions in S_k , then s_k is the degree of φ_{S_k} and is the sum of the degrees of the irreducible S_k -modules. The first few values of s_k are

k	0	1	2	3	4	5	6	7	8	9	10
s_k	1	1	2	4	10	26	76	232	764	2620	9496

(2.6)

The sequence s_k is [OEIS] Sequence A000085 and has the exponential generating function

$$e^{x^2/2+x} = \sum_{k=0}^{\infty} s_k \frac{x^k}{k!}. \quad (2.7)$$

2.2 The model representation of S_k

We now construct the corresponding induced model representation. For $t \in I_{S_k}^f$, let $M_t = \mathbb{C}t$ be the 1-dimensional $C(t)$ -module with character π_f , so that $c \in C(t)$ acts on t by $c \cdot t = S(c, t) c t c^{-1} = S(c, t)t$, where $S(c, t) = \pi_f(c)$ is the sign of the permutation induced by c on the fixed points of t . Since $C(t)$ is the stabilizer of t under the conjugation action of S_k , the cosets of $C(t)$ are in bijection with the S_k -orbits of t , which is the set of involutions $I_{S_k}^f$. Now consider the induced module

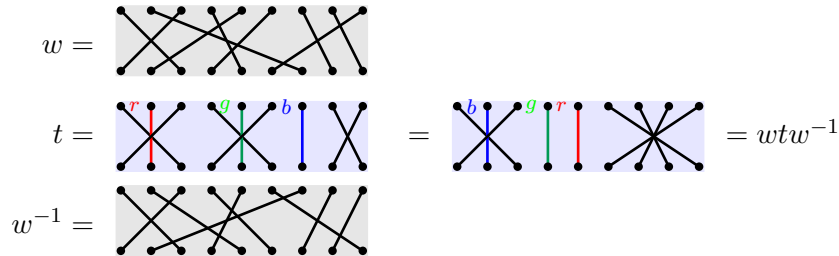
$$\text{Ind}_{C(t)}^{S_k}(M_t) \cong \mathbb{C}S_k \otimes_{C(t)} t, \quad (2.8)$$

where the S_k action is given by $x \cdot (w \otimes_{C(t)} t) = xw \otimes_{C(t)} t$, for all $w, x \in S_k$, and is extended linearly to $\mathbb{C}S_k \otimes_{C(t)} M_t$. Since M_t is 1-dimensional, $\dim(\text{Ind}_{C(t)}^{S_k}(M_t)) = |S_k|/|C(t)| = |I_{S_k}^f|$. Let $\{w_s | s \in I_{S_k}^f\}$ be a set of distinct coset representatives of $C(t) \in S_k$ such that $w_s t w_s^{-1} = s$. If $w \in S_k$ with $w = w_s c \in w_s C(t)$, then since the tensor product is over $C(t)$, we have $w \otimes_{C(t)} t = w_s c \otimes_{C(t)} t = w_s \otimes_{C(t)} c \cdot t = S(c, t) w_s \otimes_{C(t)} t$. Thus, the vectors of the form $w_s \otimes_{C(t)} t$ span $\text{Ind}_{C(t)}^{S_k}(M_t)$ and by comparing dimensions $\{w_s \otimes_{C(t)} t \mid s \in I_{n,f}\}$ is a basis of $\text{Ind}_{C(t)}^{S_k}(M_t)$.

The induced module $\text{Ind}_{C(t)}^{S_k}(M_t)$ has a nice combinatorial construction. If $w \in S_k$ and $t \in I_{n,f}$ then $wtw^{-1} \in I_{S_k}^f$ is an involution with the same number f of fixed points as t . However, the relative position of the fixed points are permuted in the map $t \mapsto wtw^{-1}$. Define $S(w, t)$ to be the sign of the permutation induced on the fixed points of t under conjugation. That is,

$$S(w, t) = (-1)^{|\{1 \leq i < j \leq k \mid t(i)=i, t(j)=j, \text{ and } w(i) > w(j)\}|}. \quad (2.9)$$

For example, when the following involution is conjugated,



the three fixed points (r, g, b) are permuted to (b, g, r) which is an induced permutation of sign -1 .

Now, define an action of $w \in S_k$ on $t \in I_{S_k}^f$ by

$$w \cdot t = S(w, t) w t w^{-1} \quad (2.10)$$

which we refer to as *signed conjugation*. Inside of the group algebra $\mathbb{C}S_k$ define

$$M_{S_k}^f = \mathbb{C}\text{-span} \left\{ t \mid t \in I_{S_k}^f \right\}, \quad (2.11)$$

and let S_k act on $M_{S_k}^f$ by extending the action of (2.10) linearly.

Proposition 2.12. *For $f = k - 2\ell$ with $0 \leq \ell \leq \lfloor k/2 \rfloor$, we have $M_{S_k}^f \cong \text{Ind}_{C(t)}^{S_k}(M_t)$.*

Proof. Let $s_i = (i, i+1)$ denote the simple transposition (given here in cycle notation) that exchanges i and $i+1$. Then s_1, \dots, s_{k-1} generate S_k , and the length $\ell(w)$ of $w \in S_k$ is the minimum number of simple transpositions needed to write w as a product of simple transpositions. Consider the coset $wC(t)$ in S_k and let $wtw^{-1} = s$. We claim that if w is of minimal length among all permutations in $wC(t)$, then under the map $t \mapsto wtw^{-1} = s$ the relative position of the fixed points of t is not changed. This can be readily verified from the diagram calculus: the length $\ell(w)$ is the number of crossings in the permutation diagram of w , and thus the permutation of minimal length that conjugates t to s does not exchange any of the fixed points of t . Now, for $s \in I_{S_k}^f$ let w_s be the unique minimal-length coset representative such that $w_s t w_s^{-1} = s$. Then $\{w_s \mid s \in I_{S_k}^f\}$ is a set of distinct coset representatives of $C(t) \in S_k$.

We now show that $x \in S_k$ acts on the basis $\{w_s \otimes_{C(t)} t \mid s \in I_{S_k}^f\}$ of $\text{Ind}_{C(t)}^{S_k}(M_t)$ identically to the way that x acts on the basis $\{s \in I_{S_k}^f\}$ of $M_{S_k}^f$. We know that $xw_s \in w_r C(t)$ for some $r \in I_{S_k}^f$ so $xw_s = w_r c$ for $c \in C(t)$, and thus

$$x \cdot (w_s \otimes_{C(t)} t) = xw_s \otimes_{C(t)} t = w_r c \otimes_{C(t)} t = w_r \otimes_{C(t)} c \cdot t = S(c, t)(w_r \otimes_{C(t)} t).$$

Now observe that $x = w_r c w_s^{-1}$ so

$$x s x^{-1} = (w_r c w_s^{-1})(w_s t w_s^{-1})(w_s c^{-1} w_r^{-1}) = w_r (c t c^{-1}) w_r^{-1} = w_r t w_r^{-1} = s.$$

Furthermore, since w_r does not change the relative order of the fixed points of t , the only place where the relative order of the fixed points of t was changed was in the conjugation $c t c^{-1} = t$. Thus $S(x, s) = S(c, t)$ and so $x \cdot s = S(x, s) x s x^{-1}$. \square

Now, define

$$M_{S_k} = \mathbb{C}\text{-span} \{ t \mid t \in I_k \} = \bigoplus_{\ell=0}^{\lfloor n/2 \rfloor} M_{S_k}^{k-2\ell}. \quad (2.13)$$

By applying Proposition 2.12 to each sum in (2.13) and using (2.4) gives

Theorem 2.14. *Under signed conjugation (3.19), the module M_{S_k} decomposes into irreducible S_k modules as $M_{S_k} = \bigoplus_{\lambda \vdash n} S_k^\lambda$.*

2.3 Comparison with other Gelfand models for S_k

Adin, Postnikov, and Roichman [APR] (and also [KM]) study a slightly different combinatorial model for S_k . Their sign is computed on the two cycles of $t \in I_k^f$ as follows:

$$s(w, t) = (-1)^{|\{1 \leq i < j \leq k \mid t(i)=j, t(j)=i, \text{ and } w(i) > w(j)\}|}. \quad (2.15)$$

and the action of S_k on I_k^f is given as

$$w \cdot t = s(w, t) w t w^{-1}, \quad \text{for } w \in S_k \text{ and } t \in I_k^f.$$

We let \overline{M}_k^f denote the corresponding S_k module, and let $\overline{M}_k = \sum_{\ell=0}^{\lfloor k/2 \rfloor} \oplus \overline{M}_k^{k-2\ell}$. In [APR] it is proved that \overline{M}_k is a Gelfand model for S_k . Moreover, the action is given a q -extension in [APR] to a Gelfand model for the Iwahori-Hecke algebra $H_k(q)$ of S_k . In what follows we prove that the Adin-Postnikov-Roichman model differs from the Saxl model by the sign representation.

Proposition 2.16. *For each $0 \leq f \leq k$ such that $k - f$ is even, we have $M_{S_k}^f \cong \overline{M}_{S_k}^f \otimes S_k^{\text{sign}}$, where S_k^{sign} is the sign representation of S_k .*

Proof. Let $t \in I_{S_k}^f$ and let $w \in S_k$ such that $wtw^{-1} = t$. That is, the t - t entry of the matrix of w is nonzero (in both $\overline{M}_{S_k}^f$ and $M_{S_k}^f$) and thus contributes to the trace. Let F be the set of fixed points of t and let $t = (a_1, b_1)(a_2, b_2) \cdots (a_\ell, b_\ell)$ be the decomposition of t into $\ell = (k - f)/2$ disjoint 2-cycles with $a_i < b_i$ for each i . Thus the complement of F in $\{1, 2, \dots, k\}$ is $\bar{F} = \{a_1, b_1, a_2, b_2, \dots, a_\ell, b_\ell\}$.

Since $wtw^{-1} = t$, we know that w permutes the fixed points F of t . Furthermore, w permutes the transpositions among themselves and possibly transposes the endpoints of the transpositions. We factor w according to these three actions. Let $w_a, w_b, w_\pi \in S_k$ be defined as follows:

1. $w_b(c) = c$ if $c \in \bar{F}$ and $w_b(d) = w(d)$ if $d \in F$; thus, w_b permutes the fixed points of t as in w and fixes the others.
2. $w_a(d) = d$ if $d \in F$, $w_a(a_i) = b_i$ and $w_a(b_i) = a_i$ if $w(a_i) > w(b_i)$, and $w_a(a_i) = a_i$ and $w_a(b_i) = b_i$ if $w(a_i) < w(b_i)$; thus w_a transposes the endpoints of the transpositions in t if they are transposed by w .
3. $w_\pi(d) = d$ if $d \in F$ and $w_\pi(a_i) = a_{\pi(i)}$ and $w_\pi(b_i) = b_{\pi(i)}$ where π is the permutation on the transpositions induced by w .

It is easy to check, by examining the values of these permutations on each element of $F \cup \bar{F} = \{1, \dots, k\}$, that

$$w = w_\pi w_a w_b, \quad \text{and thus} \quad \text{sign}(w) = \text{sign}(w_\pi) \text{sign}(w_a) \text{sign}(w_b).$$

Furthermore, by the definition of $w_a, w_b, S(w, t)$, and $s(w, t)$ we have $\text{sign}(w_b) = S(w, t)$ and $\text{sign}(w_a) = s(w, t)$. Finally, since w_π permutes the set of transpositions (a_i, b_i) , keeping $a_i < b_i$, it can be written as a product of pairs of transpositions of the form $(a_i, a_j)(b_i, b_j)$. Thus, $\text{sign}(w_\pi) = 1$, and we have $\text{sign}(w) = S(w, t)s(w, t)$ or, equivalently, $S(w, t) = s(w, t)\text{sign}(w)$, for each t such that $wtw^{-1} = t$. It follows that the characters of w on $M_{S_k}^f$ and $\overline{M}_{S_k}^f \otimes S_k^{\text{sign}}$ are equal and the modules are isomorphic. \square

3 Gelfand Models from the Jones Basic Construction

Each of the diagram algebras of interest in this paper has an inductive structure coming from a Jones basic construction ideal $A_k e_k A_k \subseteq A_k$ (described below) and a semisimple quotient $A_k / (A_k e_k A_k) \cong C_k$. We use this general structure to uniformly construct a Gelfand model for A_k by lifting those from $C_r, 0 \leq r \leq k$, to A_k . We describe here the tools necessary for our construction and point the reader to [GHJ], [Jo1], [HR1], [HR2], [GG] for further details on the Jones basic construction.

3.1 The Jones basic construction

Let A_k be the partition algebra or one of its subalgebras described in Section 1 with the parameter $x \in \mathbb{C}$ chosen such that A_k is semisimple. Let \mathcal{A}_k be the basis of diagrams which span A_k . We have a natural embedding that forms a tower of algebras,

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots, \quad (3.1)$$

where A_{k-1} is embedded as subalgebra of A_k by placing an identity edge to the right of any diagram in A_{k-1} . Let $J_{k-1} \subseteq A_k$ be the ideal spanned by the diagrams of A_k having rank $k-1$ or less. Then,

$$A_k \cong J_{k-1} \oplus C_k, \quad (3.2)$$

where C_k is the span of the diagrams of rank exactly equal to k . In the examples of this paper,

$$\begin{aligned} C_k &\cong \mathbb{C}S_k, & \text{when } A_k \text{ is one of the nonplanar algebras } P_k(x), B_k(x), RB_k(x) \text{ or } R_k, \\ C_k &\cong \mathbb{C}1_k, & \text{when } A_k \text{ is one of the planar algebras } TL_k(x), M_k(x), \text{ or } PR_k. \end{aligned} \quad (3.3)$$

In what follows, we will lift model representations from $C_r, 0 \leq r \leq k$, to a model for A_k .

Define the following element $e_k \in J_{k-1}$,

$$\begin{aligned} (a) \quad e_k &= \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & & & k-1 & k \\ \bullet & \bullet & \cdots & \bullet & \bullet & \bullet \end{array} \end{array}, & \text{for } A_k \text{ equal to } P_k(x), RB_k(x), R_k, M_k(x), \text{ or } PR_k, \\ (b) \quad e_k &= \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & & & k-1 & k \\ \bullet & \bullet & \cdots & \bullet & \bullet & \bullet \end{array} \end{array}, & \text{for } A_k \text{ equal to } B_k(x) \text{ or } TL_k(x). \end{aligned} \quad (3.4)$$

In either case $e_k^2 = xe_k$, so e_k is an essential idempotent. Recall that in the special cases where $A_k = R_k$ or $A_k = PR_k$ we have $x = 1$. It is easy to verify by diagram multiplication that

$$\begin{aligned} J_{k-1} &= A_k e_k A_k, & \text{for } A_k \text{ equal to } P_k(x), RB_k(x), R_k, M_k(x), \text{ or } PR_k, \\ J_{k-2} = J_{k-1} &= A_k e_k A_k, & \text{for } A_k \text{ equal to } B_k(x) \text{ or } TL_k(x). \end{aligned} \quad (3.5)$$

If A_k equals $P_k(x), RB_k(x), R_k, M_k(x)$, or PR_k , then A_{k-1} commutes with e_k in A_k , and we have an isomorphism $A_{k-1} \xrightarrow{\cong} e_k A_k e_k$ given by $a \mapsto ae_k = e_k a$, for any $a \in A_{k-1}$. If A_k is equal to $B_k(x)$ or $TL_k(x)$, then A_{k-2} commutes with e_k in A_k , and we have an isomorphism $A_{k-2} \xrightarrow{\cong} e_k A_k e_k$ given by $a \mapsto ae_k = e_k a$, for any $a \in A_{k-2}$. To handle the two cases simultaneously, define

$$A'_k = \begin{cases} A_{k-1}, & \text{if } A_k \text{ equals } P_k(x), RB_k(x), R_k, M_k(x), \text{ or } PR_k, \\ A_{k-2}, & \text{if } A_k \text{ equals } B_k(x) \text{ or } TL_k(x). \end{cases} \quad (3.6)$$

Now view $A_k e_k$ as a module for $A_k e_k A_k$ by multiplication on the left and as a module for $e_k A_k e_k$ by multiplication on the right, then these actions commute and centralize each other. That is,

$$J_{k-1} \cong A_k e_k A_k \cong \text{End}_{e_k A_k e_k}(A_k e_k) \quad \text{and} \quad A'_k \cong e_k A_k e_k \cong \text{End}_{A_k e_k A_k}(A_k e_k).$$

It follows from general properties of double centralizer theory (see [CR, Secs. 3B and 68] or [HR2]) that the irreducible components of J_{k-1} and A'_k have the same index set.

Let Λ_{A_k} index the irreducible representations of the semisimple algebra A_k . Then by (3.2), $\Lambda_{A_k} = \Lambda_{J_{k-1}} \sqcup \Lambda_{C_k}$ and by the previous paragraph $\Lambda_{J_{k-1}} = \Lambda_{A'_k}$, so we have

$$\Lambda_{A_k} = \Lambda_{A'_k} \sqcup \Lambda_{C_k}. \quad (3.7)$$

By induction on (3.7),

$$\begin{aligned}\Lambda_{A_k} &= \bigsqcup_{r=0}^k \Lambda_{C_r}, & \text{for } A_k \text{ equal to } P_k(x), RB_k(x), R_k, M_k(x), \text{ or } PR_k, \\ \Lambda_{A_k} &= \bigsqcup_{\ell=0}^{\lfloor k/2 \rfloor} \Lambda_{C_{k-2\ell}}, & \text{for } A_k \text{ equal to } B_k(x) \text{ or } TL_k(x).\end{aligned}\quad (3.8)$$

In Section 4, we will see this property realized for our examples. Let A_k^λ denote the irreducible A_k -module corresponding to $\lambda \in \Lambda_{A_k}$, and let $\chi_{A_k}^\lambda$ denote its corresponding character.

For any diagram $d \in \mathcal{A}_k$, we have $e_k d e_k = \varepsilon(d) e_k = e_k \varepsilon(d)$ for a unique diagram $\varepsilon(d) \in A'_k$. This extends to a map $\varepsilon : A_k \rightarrow A'_k$, called the *conditional expectation*. If χ is any character of A_k , then by (3.2), χ is completely determined by its values on C_k and J_{k-1} . If $a \in J_{k-1} = A_k e_k A_k$, then $a = a_1 e_k a_2$ for $a_1, a_2 \in A_k$, and by the trace property of χ ,

$$\chi(a) = \chi(a_1 e_k a_2) = \chi(a_2 a_1 e_k) = \frac{1}{x} \chi(a_2 a_1 e_k^2) = \frac{1}{x} \chi(e_k a_2 a_1 e_k) = \frac{1}{x} \chi(\varepsilon(a_2 a_1) e_k),$$

and thus character values on J_k are completely determined by their values on $a e_k$ for $a \in A'_k$. It follows that

$$\begin{aligned}&\text{Characters of } A_k \text{ are completely determined by their values on} \\ &b \in C_k \text{ and } a e_k, \text{ for } a \in A'_k.\end{aligned}\quad (3.9)$$

The following result is proved in [HR1] using general properties of the Jones basic construction in the case when e_k is defined as in (3.4.b) and $A'_k = A_{k-2}$. It is adapted to the case when e_k is defined as in (3.4.a) and $A'_k = A_{k-1}$ in [Ha1]. If $\lambda \in \Lambda_{A_k}$, then

$$\chi_{A_k}^\lambda(d) = \begin{cases} \chi_{C_k}^\lambda(d), & \text{if } \lambda \in \Lambda_{C_k} \text{ and } \text{rank}(d) = k, \\ 0, & \text{if } \lambda \in \Lambda_{C_k} \text{ and } \text{rank}(d) < k, \\ x \chi_{A'_k}^\lambda(a), & \text{if } \lambda \in \Lambda_{A'_k} \text{ and } d = a e_k \text{ with } a \in A'_k. \end{cases} \quad (3.10)$$

The remaining character values, $\chi_{A_k}^\lambda(d)$ for $\lambda \in \Lambda_{A'_k}$ and $d \in C_k$, are harder to compute, but they are not needed for our work in this paper. Formulas for these values are given in [Ra] and [HR1] for $B_k(x)$, in [Ha1] for $P_k(x)$, in [DHP] for R_k , in [HR1] for $TL_k(x)$, in [FHH] for PR_k , and in [BH] for $M_k(x)$. These values have not been computed for $RB_k(x)$.

3.2 Symmetric diagrams and diagram conjugation

For any diagram $d \in \mathcal{A}_k$, let $d^T \in \mathcal{A}_k$ be the diagram obtained by reflecting d over its horizontal axis. For example,

$$\begin{aligned}d_1 &= \begin{array}{c} \text{Diagram 1} \end{array} \Rightarrow d_1^T = \begin{array}{c} \text{Diagram 1}^T \end{array}, \\ d_2 &= \begin{array}{c} \text{Diagram 2} \end{array} \Rightarrow d_2^T = \begin{array}{c} \text{Diagram 2}^T \end{array}.\end{aligned}$$

Note that the map $d \rightarrow d^T$ corresponds to exchanging $i \leftrightarrow i'$ for all i .

We say that a diagram d is *symmetric* if $d^T = d$. In our example above, d_2 is symmetric and d_1 is not. If we let $(i')' = i$ and let $B' = \{b' \mid b \in B\}$ for a block B of a partition diagram d , then d is symmetric if it satisfies: $B \in d$ if and only if $B' \in d$. If d is a partition diagram, then we say that a block $B \in d$ is a *fixed block* if $B' = B$. In our above examples, d_1 has one fixed block, $\{5, 5'\}$, and d_2 has two fixed blocks, $\{8, 8'\}$ and $\{6, 7, 10, 6', 7', 10'\}$. Note that

$$(ab)^T = b^T a^T \quad \text{and} \quad (dtd^T)^T = (d^T)^T t^T d^T = dtd^T,$$

so t is symmetric if and only if dtd^T is symmetric. We say that dtd^T is the *conjugate* of t by d .

Remark 3.11. The symmetric diagrams in the partition algebra are referred to as type-B set partitions in [OEIS] Sequence A002872). They are also closely related to the type-B set partitions used in [Re], except that in [Re] the partitions are restricted to have at most one fixed block.

Remark 3.12. If we restrict our diagrams to the symmetric group, then d^T equals d^{-1} , diagram conjugation corresponds to usual group conjugation, symmetric diagrams are involutions, and fixed blocks are fixed points.

For $d \in \mathcal{A}_k$, let $\tau(d)$ be the set partition of $\{1, \dots, k\}$ given by restricting d to $\{1, \dots, k\}$ and let $\beta(d)$ be the set partition of $\{1', \dots, k'\}$ given by restricting d to $\{1', \dots, k'\}$. Observe that if t is symmetric, then $i \leftrightarrow i'$ gives a bijection between $\tau(t)$ and $\beta(t)$.

For any of our diagram algebras \mathcal{A}_k , define

$$\begin{aligned} \mathcal{I}_{\mathcal{A}_k}^{r,f} &= \{ d \in \mathcal{A}_k \mid d \text{ is symmetric, } \text{rank}(d) = r, \text{ and } d \text{ has } f \text{ fixed blocks} \}, \\ \mathcal{I}_{\mathcal{A}_k}^r &= \{ d \in \mathcal{A}_k \mid d \text{ is symmetric, } \text{rank}(d) = r \}, \\ \mathcal{I}_{\mathcal{A}_k} &= \{ d \in \mathcal{A}_k \mid d \text{ is symmetric} \}. \end{aligned} \quad (3.13)$$

Proposition 3.14. For any $t \in \mathcal{I}_{\mathcal{A}_k}^{r,f}$, we have

$$\mathcal{I}_{\mathcal{A}_k}^{r,f} = \{ d \circ t \circ d^T \mid d \in \mathcal{A}_k \text{ and } \text{rank}(d \circ t \circ d^T) = \text{rank}(t) \}.$$

Proof. If the rank is preserved by d , then every propagating block in t must connect to a propagating block in d , and since t is symmetric the same holds for d^T . Hence, $\text{rank}(d \circ t \circ d^T) = \text{rank}(t) = r$. Furthermore, since fixed blocks propagate, every fixed block of t must connect to a propagating edge in d on top and d^T on bottom. Since d^T is the horizontal reflection of d , the corresponding block in $d \circ t \circ d^T$ is fixed. The same argument holds for any non-fixed propagating block. Thus, if $\text{rank}(d \circ t \circ d^T) = \text{rank}(t)$, then $d \circ t \circ d^T \in \mathcal{I}_{\mathcal{A}_k}^{r,f}$.

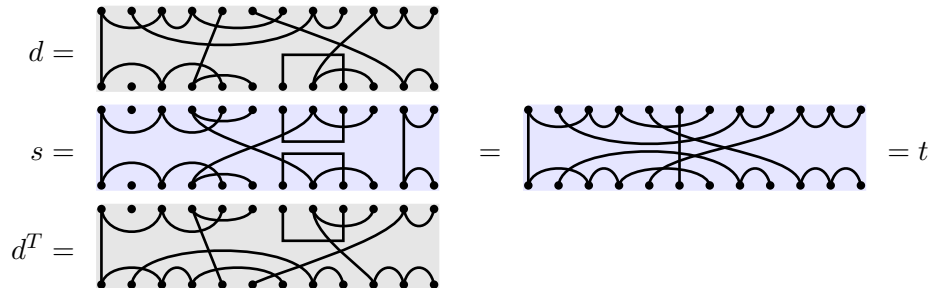
Now, for any $s, t \in \mathcal{I}_{\mathcal{A}_k}^{r,f}$, we construct $d \in \mathcal{A}_k$ such that $d \circ s \circ d^T = t$ as follows:

- (i) Partition the top and bottom row of d so that $\tau(d) = \tau(t) = \beta(t)$ and $\beta(d) = \tau(s) = \beta(s)$.
- (ii) In order from left to right, connect the fixed blocks of s in $\tau(d)$ to the fixed blocks of t in $\beta(d)$.
- (iii) In order from left to right, connect the non-fixed propagating blocks of s in $\tau(d)$ to the non-fixed propagating blocks of t in $\beta(s)$.

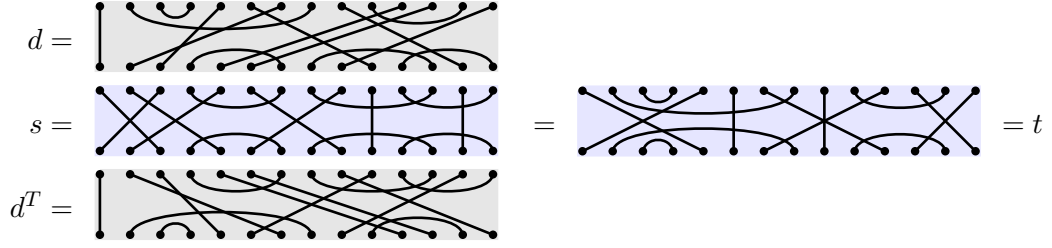
From this construction we have $d \circ s \circ d^T = t$, since we have moved the propagating blocks of s to the propagating blocks of t and we have moved the fixed blocks of s to the fixed blocks of t . We illustrate this process in Example 3.15 below. \square

Example 3.15. Examples of diagram conjugation, $dsd^T = t$.

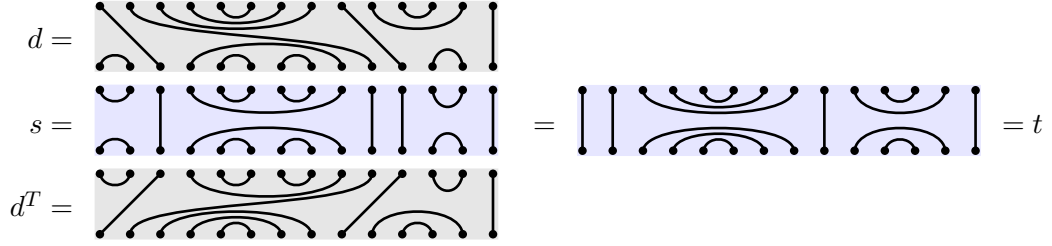
(a) Conjugation in the partition algebra $\mathcal{P}_k(x)$. Observe that the sizes of the blocks can change as they get permuted.



(b) Conjugation in the Brauer algebra $B_k(x)$:



(c) Conjugation in the Temperley-Lieb algebra $TL_k(x)$:



3.3 A Gelfand model representation for A_k

For $0 \leq f \leq r \leq k$, define

$$M_{A_k}^{r,f} = \mathbb{C}\text{-span}\{d \mid d \in I_{A_k}^{r,f}\}, \quad (3.16)$$

where $M_{A_k}^{r,f} = 0$ if $I_{A_k}^{r,f} = \emptyset$, and let

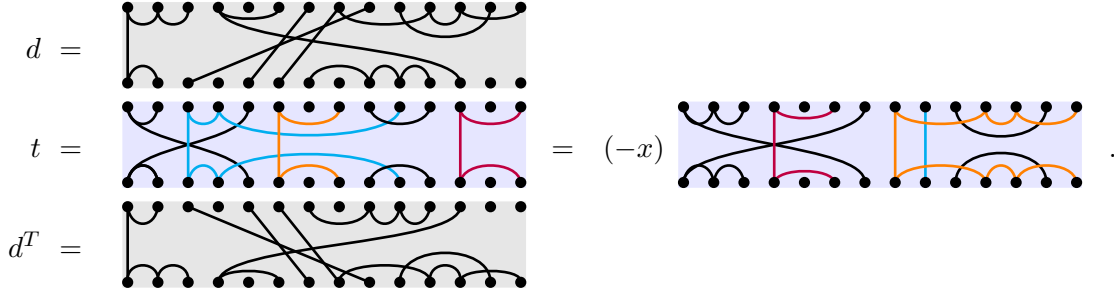
$$\begin{aligned} M_{A_k}^r &= \mathbb{C}\text{-span}\{d \mid d \in I_{A_k}^r\}, & M_{A_k} &= \mathbb{C}\text{-span}\{d \mid d \in I_{A_k}\}, \\ &= \bigoplus_{f=0}^r M_{A_k}^{r,f}, & \text{and} & &= \bigoplus_{r=0}^k M_{A_k}^r = \bigoplus_{r=0}^k \bigoplus_{f=0}^r M_{A_k}^{r,f}. \end{aligned} \quad (3.17)$$

If $d \in \mathcal{A}_k$ and $t \in I_{A_k}^{r,f}$, then there are two possibilities for the map $t \mapsto d \circ t \circ d^T$. Either $\text{rank}(d \circ t \circ d^T) < \text{rank}(t)$ or $\text{rank}(d \circ t \circ d^T) = \text{rank}(t)$. In the later case, the fixed blocks of t have been permuted, and we let $S(d, t)$ be the sign of the permutation of the fixed blocks of t . For $d \in \mathcal{A}_k$ and $t \in I_{A_k}^{r,f}$, define

$$d \cdot t = \begin{cases} x^{\kappa(d,t)} S(d, t) d \circ t \circ d^T, & \text{if } \text{rank}(d \circ t \circ d^T) = \text{rank}(t), \\ 0, & \text{if } \text{rank}(d \circ t \circ d^T) < \text{rank}(t), \end{cases} \quad (3.18)$$

where $\kappa(d, t)$ is the number of blocks (1.1) removed from the middle row in creating $d \circ t$. We refer to the above action as *signed conjugation* of t by d .

Example 3.19. (Signed Conjugation) *In the following example, there is one block removed in $d \circ t$ yielding a multiple of x . Furthermore, the three fixed blocks of t are permuted as $(B_1, B_2, B_3) \mapsto (B_3, B_2, B_1)$. Hence, the sign is $S(d, t) = -1$.*



Remark 3.20. Observe the following subtlety in the definition of this action: as a product in A_k we have $dtd^T = x^{2\kappa(d,t)}d \circ t \circ d^T$, since each block removed from the middle row in dt has a mirror image in td^T ; however, we require $d \cdot t = x^{\kappa(d,t)}S(d,t)d \circ t \circ d^T$ in order to make this an algebra action, as will be seen in the proof of the next proposition.

Remark 3.21. When the action in (3.18) is restricted to the symmetric group, we exactly get the action of S_k on involutions I_k defined in equation (2.10)

Proposition 3.22. The action defined in (3.18) makes $M_{A_k}^{r,f}$ an A_k -module.

Proof. We must show that $(d_1 d_2) \cdot t = d_1 \cdot (d_2 \cdot t)$. If $\text{rank}(d \circ t \circ d^T) < \text{rank}(t)$, then by the associativity of diagram multiplication we have $(d_1 d_2) \cdot t = 0 = d_1 \cdot (d_2 \cdot t)$. So we assume that $\text{rank}(d \circ t \circ d^T) = \text{rank}(t)$. Let $d_1 \circ d_2 = d_3$ and let $d_2 \circ t \circ d_2^T = s$ for some $s \in I_{A_k}^{r,f}$. Then we have,

$$\begin{aligned}
 d_1 \cdot (d_2 \cdot t) &= x^{\kappa(d_2,t)}S(d_2,t)d_1 \cdot (d_2 \circ t \circ d_2^T) \\
 &= x^{\kappa(d_1,s)}x^{\kappa(d_2,t)}S(d_1,s)S(d_2,t)(d_1 \circ (d_2 \circ t \circ d_2^T) \circ d_1^T) \\
 &= x^{\kappa(d_1,s)}x^{\kappa(d_2,t)}S(d_1,s)S(d_2,t)((d_1 d_2) \circ t \circ (d_1 \circ d_2)^T) \\
 &= x^{\kappa(d_1,s)}x^{\kappa(d_2,t)}S(d_1,s)S(d_2,t)(d_3 \circ t \circ d_3^T).
 \end{aligned}$$

On the other hand,

$$(d_1 d_2) \cdot t = x^{\kappa(d_1,d_2)}d_3 \cdot t = x^{\kappa(d_1,d_2)}x^{\kappa(d_3,t)}S(d_3,t)d_3 t d_3^T,$$

so it suffices to show that

$$S(d_3,t) = S(d_1,s)S(d_2,t) \quad \text{and} \quad x^{\kappa(d_1,d_2)}x^{\kappa(d_3,t)} = x^{\kappa(d_1,s)}x^{\kappa(d_2,t)}.$$

From the diagram calculus, we have that $\kappa(d_1,d_2) = \kappa(d_1,s)$ and $\kappa(d_3,t) = \kappa(d_2,t)$, so the second equality follows immediately. The first equality corresponds to the composition of permutations of fixed blocks, and the result follows from the symmetric group fact that the sign of a permutation is multiplicative. \square

For each $t \in I_{A_k}^{r,f}$, let $p_t \in \mathcal{A}_k$ be the unique diagram such that

- (a) $\tau(p_t) = \tau(t)$ and $\beta(p_t) = \beta(t)$
- (b) A block of t is propagating if and only if the corresponding block of p_t is an identity block.

For example,

$$\begin{array}{lcl} t & = & \text{diagram 1} \\ p_t & = & \text{diagram 2} \end{array}$$

It follows easily from this construction that

$$p_t t = t p_t = x^\ell t, \quad (3.23)$$

where ℓ is the number of non-propagating blocks of t . Observe also that $\text{rank}(p_t) = \text{rank}(t)$. These diagrams are used in the proof of the following proposition.

Proposition 3.24. *If $r \neq s$, there is no submodule of $M_{A_k}^{r,f}$ isomorphic to a submodule of $M_{A_k}^{s,g}$.*

Proof. By Schur's lemma, $M_{A_k}^{r,f}$ and $M_{A_k}^{s,g}$ have an isomorphic submodule if and only if there exists a nontrivial A_k -module homomorphism $\phi : M_{A_k}^{r,f} \rightarrow M_{A_k}^{s,g}$. Assume $r < s$, without loss of generality, and let $t \in I_{A_k}^{r,f}$. Suppose that $\phi : M_{A_k}^{r,f} \rightarrow M_{A_k}^{s,g}$ is a nontrivial A_k -module homomorphism. Then by (3.23), $\phi(t) = \phi(x^{-\ell} p_t t) = x^{-\ell} p_t \phi(t)$. Now, $\phi(t)$ is a linear combination of symmetric diagrams of rank s , but $\text{rank}(p_t) = \text{rank}(t) < s$, and thus by (3.18), p_t acts as 0 on each diagram in the linear combination $\phi(t)$. Thus $\phi(t) = 0$ for each t , and there are no nontrivial A_k -module homomorphisms. \square

Let $\varphi_{A_k}^{r,f}$ be the character of the A_k -module $M_{A_k}^{r,f}$. Then

$$\varphi_{A_k}^r = \sum_{f=0}^r \varphi_{A_k}^{r,f} \quad \text{is the character of the } A_k\text{-module } M_{A_k}^r, \quad (3.25)$$

$$\varphi_{A_k} = \sum_{r=0}^k \varphi_{A_k}^r \quad \text{is the character of the } A_k\text{-module } M_{A_k}. \quad (3.26)$$

Let $\varphi_{C_k}^f$ be the restriction of $\varphi_{A_k}^{k,f}$ from A_k to C_k . Recall from (3.9) that it is sufficient to compute A_k characters on $d \in C_k$ or $d = a e_k$ with $a \in A'_k$. Compare the following formula to the analogous formula (3.10) for irreducible characters of A_k .

Proposition 3.27. *For each $d \in A_k$ and $0 \leq f \leq r \leq k$, we have*

$$\varphi_{A_k}^{r,f}(d) = \begin{cases} \varphi_{C_k}^f(d), & \text{if } r = k \text{ and } \text{rank}(d) = k, \\ 0, & \text{if } r = k \text{ and } \text{rank}(d) < k, \\ x \varphi_{A'_k}^{r,f}(a), & \text{if } r < k \text{ and } d = a e_k \text{ with } a \in A'_k. \end{cases}$$

Proof. If $r = k$ and $\text{rank}(d) < k$, then by (3.18) d acts as 0 on every $t \in I_{A_k}^{k,f}$ and thus $\varphi_{A_k}^{k,f}(d) = 0$. If $r = k$ and $\text{rank}(d) = k$, then $d \in C_k$. The restriction to diagrams of rank $r = k$ is exactly the action of C_k on $I_{A_k}^{k,f} = I_{C_k}^f$. When $C_k = \mathbb{C}S_k$ this is the Saxl representation as observed in Remark 3.21. In the planar case when $C_k = \mathbb{C}\mathbf{1}_k$, the only planar rank k diagram is $\mathbf{1}_k$ and we must have $k = f$.

Let $r < k$ and $d = a e_k = e_k a$. Then $t \in I_{A_k}^{r,f}$ contributes to the trace of d only if $d \circ t \circ d^T = t$. Furthermore, $d \circ t \circ d^T = (e_k a) \circ t \circ (e_k a)^T = e_k a \circ t \circ a^T e_k^T = e_k a \circ t \circ a^T e_k = a' e_k$ with

$a' \in \mathcal{A}_{k-1}$. Thus t contributes to the trace only if $t = t'e_k$ for $t' \in \mathcal{I}_{A'_k}^{r,f}$. Now, $d \cdot t = (ae_k) \cdot (t'e_k) = x^{\kappa(ae_k, t'e_k)} S(ae_k, t'e_k)(ae_k) \circ (t'e_k) \circ (ae_k)^T = x^{\kappa(ae_k, t'e_k)} S(ae_k, t'e_k)(a \circ t' \circ a^T)(e_k \circ e_k \circ e_k) = x^{\kappa(ae_k, t'e_k)} S(ae_k, t'e_k)(a \circ t' \circ a^T)e_k$. By examining the diagrams, and using the fact that both a and t' commute with e_k , we see that $x^{\kappa(ae_k, t'e_k)} = x^{\kappa(a, t')+1}$ and $S(ae_k, t'e_k) = S(a, t')$. Therefore, t contributes to the trace if and only if $t = t'e_k$ and, in this case, the t - t entry of the action of d on $M_{A_k}^{r,f}$ equals x times the t' - t' entry of the action of a on $M_{A'_k}^{r,f}$. Thus $\varphi_{A_k}^{r,f}(d) = x\varphi_{A'_k}^{r,f}(a)$. \square

Now assume that $\varphi_{C_k} = \sum_f \varphi_{C_k}^f$ (where the sum is over the f which are a possible number of fixed points) is a model character for C_k . This is the case for the two situations that arise in this paper, namely $C_k = \mathbb{C}S_k$ and $C_k = \mathbb{C}\mathbf{1}_k$. Let $\Lambda_{C_k}^f$ index the irreducible C_k modules which appear in $M_{C_k}^f$ so that

$$\Lambda_{C_k} = \bigsqcup_f \Lambda_{C_k}^f, \quad \text{and} \quad \varphi_{C_k}^f = \sum_{\lambda \in \Lambda_{C_k}^f} \chi_{C_k}^\lambda. \quad (3.28)$$

Theorem 3.29. *For each $0 \leq f \leq r \leq k$, we have*

$$\varphi_{A_k}^{r,f} = \sum_{\lambda \in \Lambda_{C_r}^f} \chi_{A_k}^\lambda, \quad \text{and thus} \quad \varphi_{A_k} = \sum_{r=0}^k \sum_{f=0}^r \varphi_{A_k}^{r,f} = \sum_{\lambda \in \Lambda_{A_k}} \chi_{A_k}^\lambda,$$

where $\varphi_{A_k}^{r,f} = 0$ if there do not exist symmetric diagrams of rank r with f fixed points in A_k .

Proof. The second statement follows immediately from the first. Our proof of the first statement is by induction on k , with the case $k = 0$ being trivial. Let $k > 0$ and first consider the situation where $r = k$. From Proposition 3.27 we know that $\varphi_{A_k}^{k,f}$ and $\varphi_{C_k}^f$ agree on all $d \in \mathcal{A}_k$ with $\text{rank}(d) = k$. Furthermore, by the fact that φ_{C_k} is a model for C_k and by (3.10) we have

$$\varphi_{A_k}^{k,f}(d) = \varphi_{C_k}^f(d) = \sum_{\lambda \in \Lambda_{C_k}^f} \chi_{C_k}^\lambda(d) = \sum_{\lambda \in \Lambda_{C_k}^f} \chi_{A_k}^\lambda(d), \quad \text{for all } d \text{ with } \text{rank}(d) = k.$$

Also by Proposition 3.27 and equation (3.10), we know that if $d \in \mathcal{A}_k$ with $\text{rank}(d) < k$, then $0 = \varphi_{A_k}^{k,f}(d) = \sum_{\lambda \in \Lambda_{C_k}^f} \chi_{A_k}^\lambda(d)$. Thus, $\varphi_{A_k}^{k,f} = \sum_{\lambda \in \Lambda_{C_k}^f} \chi_{A_k}^\lambda$ is the decomposition of $\varphi_{A_k}^{k,f}$ into irreducible characters.

Now let $r < k$. The previous paragraph and Proposition 3.24 tell us that the decomposition of $\varphi_{A_k}^{r,f}$ into irreducibles does not involve any of the $\chi_{A_k}^\lambda$ with $\lambda \in \Lambda_{C_k}$. Thus by (3.7) it is of the form

$$\varphi_{A_k}^{r,f} = \sum_{\lambda \in \Lambda_{A'_k}} a_\lambda \chi_{A_k}^\lambda, \quad \text{for some } a_\lambda \in \mathbb{Z}_{\geq 0}. \quad (3.30)$$

For $d = ae_k$ with $a \in A'_k$ we can apply the inductive hypothesis since $A'_k = A_{k-1}$ or A_{k-2} ,

$$\varphi_{A_k}^{r,f}(d) = x\varphi_{A'_k}^{r,f}(a) = \sum_{\lambda \in \Lambda_{C_r}^f} x\chi_{A'_k}^\lambda(a) = \sum_{\lambda \in \Lambda_{C_r}^f} \chi_{A_k}^\lambda(d).$$

Thus, $a_\lambda = 1$ for $\lambda \in \Lambda_{C_r}^f$ and $a_\lambda = 0$, otherwise, as desired. \square

When applied to the corresponding modules, Theorem 3.29 says the following.

Theorem 3.31. *For each $0 \leq f \leq r \leq k$, we have*

$$M_{A_k}^{r,f} \cong \bigoplus_{\lambda \in \Lambda_{C_r}^f} M_{A_k}^\lambda \quad \text{and thus} \quad M_{A_k} \cong \bigoplus_{r=0}^k \bigoplus_{f=0}^r M_{A_k}^{r,f} \cong \bigoplus_{\lambda \in \Lambda_{A_k}} M_{A_k}^\lambda,$$

where $M_{A_k}^{r,f} = 0$ if there do not exist symmetric diagrams of rank r with f fixed points in A_k .

In our examples when A_k is nonplanar, we have $C_r = \mathbb{C}S_r$, and by Saxl's result (2.4), $\Lambda_{C_k}^f = \{ \lambda \vdash r \mid \text{odd}(\lambda) = f \}$, where $\text{odd}(\lambda)$ is the number of odd parts of λ . If A_k is planar, then $C_r = \mathbb{C}\mathbf{1}_r$ and $M_{C_r}^{r,f}$ is irreducible and 1-dimensional if $r = f$ and is 0 if $r < f$. Thus,

Corollary 3.32. *If A_k is planar, then $M_{A_k}^r$ is irreducible, and thus $M_{A_k} = \bigoplus_{r=0}^k M_{A_k}^r$ is a decomposition into irreducible A_k -modules.*

4 Gelfand Models for Diagram Algebras

In this section we illustrate the details of our model representations for each of our diagram algebras.

4.1 The partition algebra $P_k(x)$

The partition algebra $P_k(x)$ is spanned by the set partitions \mathcal{P}_k defined in Section 1.1, has dimension equal to the Bell number $B(2k)$, and is semisimple for $x \in \mathbb{C}$ such that $x \notin \{0, 1, \dots, 2k-1\}$ (see [MS] or [HR2]). When semisimple, its irreducible representations are indexed by partitions in the set

$$\Lambda_{P_k} = \{ \lambda \vdash r \mid 0 \leq r \leq k \}. \quad (4.1)$$

Let P_k^λ denote the irreducible module indexed by $\lambda \in \Lambda_{P_k}$, and let $\chi_{P_k}^\lambda$ denote its character.

For each $0 \leq \ell \leq \lfloor r/2 \rfloor$ there exist symmetric partition algebra diagrams in $P_k^{r,f}$ of rank r with $f = r - 2\ell$ fixed blocks and ℓ blocks which are transposed (i.e., propagating, nonidentity blocks); see Example 3.15. The number of these symmetric diagrams is

$$\dim M_{P_k}^{r,r-2\ell} = |P_k^{r,r-2\ell}| = \sum_{b=r}^k S(k, b) \binom{b}{r} \binom{r}{2\ell} (2\ell - 1)!!, \quad (4.2)$$

where $S(k, b)$ is a Stirling number of the second kind. This sum is justified as follows: first partition the top and bottom rows of a symmetric diagram identically into b blocks in $S(k, b)$ ways. Then choose r of these blocks to be propagating, and from those r blocks, choose 2ℓ of them to correspond to transpositions and match them up in $(2\ell - 1)!!$ ways. The remaining $r - 2\ell$ blocks are fixed.

For the partition algebra, Theorem 3.31 becomes

$$M_{P_k}^{r,f} = \bigoplus_{\substack{\lambda \vdash k \\ \text{odd}(\lambda)=f}} P_k^\lambda \quad \text{and} \quad M_{P_k} = \bigoplus_{r=0}^k \bigoplus_{\ell=0}^{\lfloor r/2 \rfloor} M_{P_k}^{r,r-2\ell} = \bigoplus_{\lambda \in \Lambda_{P_k}} P_k^\lambda. \quad (4.3)$$

If we let $p_k = |P_k| = \sum_{r=0}^k \sum_{\ell=0}^{\lfloor r/2 \rfloor} |P_k^{r,r-2\ell}| = \dim M_{P_k}$ denote the total number of symmetric diagrams in $P_k(x)$, then p_k is the sum of the degrees of the irreducible $P_k(x)$ -modules (which can

be found in [Ma], [HR2], [Ha1]). The first few values of \mathbf{p}_k are

k	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{p}_k = \dim \mathbf{M}_{\mathbf{P}_k}$	1	2	7	31	164	999	6841	51790	428131	3827967	36738144

(4.4)

The sequence \mathbf{p}_k is [OEIS] Sequence A002872, which equals the number of type- B set partitions (see Remark 3.11), and has exponential generating function

$$e^{(e^{2x}-3)/2+e^x} = \sum_{k=0}^{\infty} \mathbf{p}_k \frac{x^k}{k!}. \quad (4.5)$$

4.2 The Brauer algebra $\mathbf{B}_k(x)$

The Brauer algebra $\mathbf{B}_k(x)$ is the subalgebra of $\mathbf{P}_k(x)$ spanned by the Brauer diagrams. It has dimension $\dim \mathbf{B}_k(x) = (2k-1)!!$ and is semisimple for $x \in \mathbb{C}$ chosen to avoid $\{x \in \mathbb{Z} \mid 4-2k \leq x \leq k-2\}$ (see [Rui]). When $\mathbf{B}_k(x)$ is semisimple, its irreducible modules are indexed by partitions in the set

$$\Lambda_{\mathbf{B}_k} = \{ \lambda \vdash (k-2r) \mid 0 \leq r \leq \lfloor k/2 \rfloor \}. \quad (4.6)$$

Let \mathbf{B}_k^λ denote the irreducible module indexed by $\lambda \in \Lambda_{\mathbf{B}_k}$, and let $\chi_{\mathbf{B}_k}^\lambda$ denote its character.

Symmetric Brauer diagrams consist of ℓ transpositions, f fixed points, and c contractions (symmetric pairs of horizontal edges). For example, the symmetric Brauer diagram,



$$\in \mathbf{B}_{14}(x)$$

has $\ell = 3$ transpositions $(1, 3), (2, 5), (6, 9)$, $c = 3$ contractions in positions $\{4, 7\}, \{8, 12\}, \{11, 14\}$, $f = 2$ fixed points in positions 10 and 13, and rank $r = 8$. Observe that these diagrams satisfy $r = k - 2c$, $0 \leq \ell \leq \lfloor (k-2c)/2 \rfloor$, and $f = k - 2c - 2\ell$, and that the number of symmetric diagrams of this type is

$$\dim \mathbf{M}_{\mathbf{B}_k}^{r,f} = \dim \mathbf{M}_{\mathbf{B}_k}^{r,r-2\ell} = |\mathbf{I}_{\mathbf{B}_k}^{r,r-2\ell}| = \binom{k}{r} (k-r-1)!! \binom{r}{2\ell} (2\ell-1)!! \quad (4.7)$$

This count is justified as follows: choose the r positions for the propagating edges in $\binom{k}{r}$ ways and pair the remaining $k-r$ positions for contractions in $(k-r-1)!!$ ways. Among the propagating edges, choose $r-2\ell$ fixed points and pair the remaining edges in transpositions in $(2\ell-1)!!$ ways.

For the Brauer algebra, Theorem 3.31 becomes

$$\mathbf{M}_{\mathbf{B}_k}^{r,f} \cong \bigoplus_{\substack{\lambda \vdash r \\ \text{odd}(\lambda)=f}} \mathbf{B}_k^\lambda \quad \text{and} \quad \mathbf{M}_{\mathbf{B}_k} \cong \bigoplus_{c=0}^{\lfloor k/2 \rfloor} \bigoplus_{\ell=0}^{\lfloor (k-2c)/2 \rfloor} \mathbf{M}_{\mathbf{B}_k}^{k-2c,k-2c-2\ell} \cong \bigoplus_{\lambda \in \Lambda_{\mathbf{B}_k}} \mathbf{B}_k^\lambda. \quad (4.8)$$

If we let $\mathbf{b}_k = |\mathbf{I}_{\mathbf{B}_k}| = \sum_{c=0}^{\lfloor k/2 \rfloor} \sum_{\ell=0}^{\lfloor (k-2c)/2 \rfloor} |\mathbf{I}_{\mathbf{B}_k}^{k-2c,k-2c-2\ell}| = \dim \mathbf{M}_{\mathbf{B}_k}$ denote the total number of symmetric diagrams in $\mathbf{B}_k(x)$, then \mathbf{b}_k is the sum of the degrees of the irreducible $\mathbf{B}_k(x)$ -modules (see [Ra]). This value can be obtained by summing (4.7) over the given values of c and ℓ or by summing over $m = c + \ell$ as follows,

$$\dim \mathbf{M}_{\mathbf{B}_k} = \sum_{m=0}^{\lfloor k/2 \rfloor} \binom{k}{2m} (2m-1)!! 2^m = \sum_{m=0}^{\lfloor k/2 \rfloor} \binom{k}{2m} \frac{(2m)!}{m!} = \sum_{m=0}^{\lfloor k/2 \rfloor} \binom{k}{2m} \binom{2m}{m} m!. \quad (4.9)$$

Here we choose $2m$ points to be the endpoints of the transpositions and contractions, we pair them up in $(2m-1)!!$ ways, and then we decide in 2^m ways if each is to be a transposition or a contraction. The first few values of \mathbf{b}_k are

k	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{b}_k = \dim \mathbf{M}_{\mathbf{R}_k}$	1	1	3	7	25	81	331	1303	5937	26785	133651

(4.10)

This sequence, \mathbf{b}_k , is [OEIS] Sequence A047974 and has exponential generating function

$$e^{x^2+x} = \sum_{k=0}^{\infty} \mathbf{b}_k \frac{x^k}{k!}.$$

4.3 The rook monoid algebra \mathbf{R}_k

The rook monoid algebra \mathbf{R}_k is the subalgebra of $\mathbf{P}_k(x)$ spanned by rook monoid diagrams with parameter set to $x = 1$. It has dimension $\dim \mathbf{R}_k = \sum_{\ell=0}^k \binom{k}{\ell}^2 \ell!$ and is semisimple (see [So], [Ha2], [KM]) with irreducible modules labeled by

$$\Lambda_{\mathbf{R}_k} = \{ \lambda \vdash r \mid 0 \leq r \leq [k] \}.$$
(4.11)

Let \mathbf{R}_k^λ denote the irreducible module labeled by $\lambda \in \Lambda_{\mathbf{R}_k}$, and let $\chi_{\mathbf{R}_k}^\lambda$ denote its character.

Symmetric rook monoid diagrams in \mathbf{R}_k consist of f fixed points, ℓ transpositions, and $k - f - 2\ell$ vertical pairs of empty vertices. For example, the symmetric rook monoid diagram,



has $\ell = 3$ transpositions $(2, 5), (7, 9), (8, 13)$, $f = 5$ fixed points 4, 6, 10, 11, 14, empty vertices in positions 1, 3, 12, and rank $r = 11$. Observe that $f = r - 2\ell$ and that every pair $0 \leq f \leq r \leq k$, with $r - f$ even, is possible, that the number of symmetric rook diagrams of this type is

$$\dim \mathbf{M}_{\mathbf{R}_k}^{r,f} = \dim \mathbf{M}_{\mathbf{R}_k}^{r,r-2\ell} = |\mathbf{I}_{\mathbf{R}_k}^{r,r-2\ell}| = \binom{k}{r} \binom{r}{2\ell} (2\ell - 1)!!, \quad (4.12)$$

This count is justified as follows: choose the r positions for propagating edges in $\binom{k}{r}$ ways, choose $r - 2\ell$ positions for fixed points among these in $\binom{r}{2\ell}$ ways, and pair the remaining propagating edges into transpositions in $(2\ell - 1)!!$ ways.

For the rook monoid algebra, Theorem 3.31 becomes

$$\mathbf{M}_{\mathbf{R}_k}^{r,f} \cong \bigoplus_{\substack{\lambda \vdash r \\ \text{odd}(\lambda)=f}} \mathbf{R}_k^\lambda \quad \text{and} \quad \mathbf{M}_{\mathbf{R}_k} \cong \bigoplus_{r=0}^k \bigoplus_{\ell=0}^{\lfloor r/2 \rfloor} \mathbf{M}_{\mathbf{R}_k}^{r,r-2\ell} \cong \bigoplus_{\lambda \in \Lambda_{\mathbf{R}_k}} \mathbf{R}_k^\lambda. \quad (4.13)$$

If we let $\mathbf{r}_k = |\mathbf{I}_{\mathbf{R}_k}| = \sum_{r=0}^k \sum_{\ell=0}^{\lfloor r/2 \rfloor} |\mathbf{I}_{\mathbf{R}_k}^{r,r-2\ell}| = \dim \mathbf{M}_{\mathbf{R}_k}$ denote the total number of symmetric diagrams in \mathbf{R}_k , then \mathbf{r}_k is the sum of the degrees of the irreducible \mathbf{R}_k -modules (which can be found in [So], [Ha2]). The first few values of these dimensions are

k	0	1	2	3	4	5	6	7	8	9	10
$\dim \mathbf{M}_{\mathbf{R}_k}$	1	2	5	14	43	142	499	1850	7193	29186	123109

(4.14)

The sequence r_k gives the number of “self-inverse partial permutations” and is [OEIS] Sequence A005425. Furthermore, r_k is related to the number of involutions s_k in the symmetric group (2.6) by the binomial transform $r_k = \sum_{i=0}^k \binom{k}{i} s_i$ and thus (see [GKP, (7.75)]) has exponential generating function

$$e^x e^{x^2/2+x} = e^{x^2/2+2x} = \sum_{k=0}^{\infty} r_k \frac{x^k}{k!}. \quad (4.15)$$

Remark 4.16. *The model representation that we construct with our methods here differs from the model for the rook monoid given in [KM] in the same way that the Saxl symmetric group model differs from the one used by Adin, Postnikov, and Roichman [APR]. See Section 2.3.*

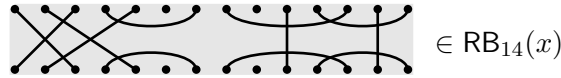
4.4 The rook-Brauer algebra $\text{RB}_k(x)$

The rook-Brauer algebra $\text{RB}_k(x)$ is the subalgebra of the partition algebra $\text{P}_k(x)$ spanned by rook-Brauer diagrams. It has dimension $\sum_{\ell=0}^k \binom{2k}{2\ell} (2\ell-1)!!$ (see [dH]). The rook-Brauer algebra $\text{RB}_k(x)$ is semisimple for all but finitely many $x \in \mathbb{C}$ (the exact values have not been determined; see [dH] or [MM]). When x is chosen so that $\text{RB}_k(x)$ is semisimple, its irreducible modules are labeled by

$$\Lambda_{\text{RB}_k} = \{ \lambda \vdash r \mid 0 \leq r \leq k \}. \quad (4.17)$$

Let RB_k^λ denote the irreducible module labeled by $\lambda \in \Lambda_{\text{RB}_k}$, and let $\chi_{\text{RB}_k}^\lambda$ denote its character.

Symmetric rook-Brauer diagrams in $\text{RB}_k(x)$ consist of ℓ transpositions, f fixed points, c contractions (symmetric pairs of horizontal edges), and $k - 2\ell - 2c - r$ vertical pairs of empty vertices. For example, the symmetric rook-Brauer diagram,



has $\ell = 2$ transpositions $(1, 3), (2, 5)$, $c = 3$ contractions in positions $\{4, 7\}, \{8, 12\}, \{11, 14\}$, $f = 2$ fixed points in positions 10 and 13, empty vertices in positions 6 and 9, and rank $r = 6$. Observe that these diagrams satisfy $f = r - 2\ell$, and that every pair $0 \leq f \leq r \leq k$, with $r - f$ even, is possible. The number of symmetric diagrams of this type is

$$\dim M_{\text{RB}_k}^{r,f} = \dim M_{\text{RB}_k}^{r,r-2\ell} = |\text{RB}_k^{r,r-2\ell}| = \sum_{c=0}^{\lfloor (k-r)/2 \rfloor} \binom{k}{r} \binom{k-r}{2c} (2c-1)!! \binom{r}{2\ell} (2\ell-1)!!, \quad (4.18)$$

where here we sum over the number c of contractions. This count is justified as follows: in $\binom{k}{r}$ ways, choose r positions for the propagating edges. Then from the non propagating points, select the $2c$ endpoints for the contractions in $\binom{k-r}{2c}$ ways and match them up in $(2c-1)!!$ ways. Then choose the 2ℓ endpoints of the transpositions in $\binom{r}{2\ell}$ ways, and match them up in $(2\ell-1)!!$ ways.

For the rook-Brauer algebra, Theorem 3.31 becomes

$$M_{\text{RB}_k}^{r,f} \cong \bigoplus_{\substack{\lambda \vdash r \\ \text{odd}(\lambda)=f}} \text{RB}_k^\lambda \quad \text{and} \quad M_{\text{RB}_k} \cong \bigoplus_{r=0}^k \bigoplus_{\ell=0}^{\lfloor r/2 \rfloor} M_{\text{RB}_k}^{r,r-2\ell} \cong \bigoplus_{\lambda \in \Lambda_{\text{RB}_k}} \text{RB}_k^\lambda. \quad (4.19)$$

If we let $\text{rb}_k = |\text{RB}_k| = \sum_{r=0}^k \sum_{\ell=0}^{\lfloor r/2 \rfloor} |\text{RB}_k^{r,r-2\ell}| = \dim M_{\text{RB}_k}$ denote the total number of symmetric diagrams in $\text{RB}_k(x)$, then rb_k is the sum of the degrees of the irreducible $\text{RB}_k(x)$ -modules (these dimensions can be found in [dH] or [MM]). The first few values of rb_k are

k	0	1	2	3	4	5	6	7	8	9	10
$\text{rb}_k = \dim M_{\text{RB}_k}$	1	2	6	20	76	312	1384	6512	32400	168992	921184

(4.20)

The sequence \mathbf{rb}_k is [OEIS] Sequence A000898 and it is related to the number of symmetric diagrams \mathbf{b}_k in the Brauer algebra (4.10) by the binomial transform $\mathbf{rb}_k = \sum_{i=0}^k \binom{k}{i} \mathbf{b}_i$ and thus (see [GKP, (7.75)]) has exponential generating function

$$e^x e^{x^2+x} = e^{x^2+2x} = \sum_{k=0}^{\infty} \mathbf{rb}_k \frac{x^k}{k!}. \quad (4.21)$$

4.5 The Temperley-Lieb algebra $\mathbf{TL}_k(x)$

The Temperley-Lieb algebra $\mathbf{TL}_k(x)$ is the subalgebra of the partition algebra $\mathbf{P}_k(x)$ spanned by the Temperley-Lieb diagrams (planar Brauer diagrams). Temperley-Lieb diagrams correspond to planar matchings of $\{1, \dots, k, 1', \dots, k'\}$, and $\mathbf{TL}_k(x)$ has dimension equal to the Catalan number $C_k = \frac{1}{k+1} \binom{2k}{k}$. The Temperley-Lieb algebra $\mathbf{TL}_k(x)$ is semisimple for $x \in \mathbb{C}$ chosen such that x is not the root of the polynomial $U_k(x/2)$ where U_k is the Chebyshev polynomial of the second kind (see [We] or [BH]). If, we let $x \in \mathbb{C}$ be chosen so that $\mathbf{TL}_k(x)$ is semisimple, then the irreducible modules are indexed by the following set of integers

$$\Lambda_{\mathbf{TL}_k} = \{ k - 2\ell \mid 0 \leq \ell \leq \lfloor k/2 \rfloor \}, \quad (4.22)$$

Let $\mathbf{TL}_k^{(r)}$ denote the irreducible module labeled by $r \in \Lambda_{\mathbf{TL}_k}$, and let $\chi_{\mathbf{TL}_k}^{(r)}$ denote its character.

Symmetric Temperley-Lieb diagrams in $\mathbf{TL}_k(x)$ consist of f fixed points, and c contractions. For example, the symmetric Temperley-Lieb diagram,



$$\in \mathbf{TL}_{14}(x)$$

has $c = 5$ contractions in positions $\{1, 2\}, \{4, 9\}, \{5, 6\}, \{7, 8\}, \{12, 13\}$, $f = 4$ fixed points in positions 3, 10, 11, and 14, and rank $r = 4$. Observe that in a Temperley-Lieb diagram $r = f = k - 2\ell$. The number of symmetric Temperley-Lieb diagrams is given by

$$\dim \mathbf{M}_{\mathbf{TL}_k}^{r,f} = |\mathbf{I}_{\mathbf{TL}_k}^{k-2\ell}| = \left\{ \begin{matrix} k \\ \ell \end{matrix} \right\} := \binom{k}{\ell} - \binom{k}{\ell-1} \quad (4.23)$$

where $\binom{a}{b}$ is the binomial coefficient. This fact is proved in [We, p. 545] or [Jo1, Sec. 5.1].

For the Temperley-Lieb algebra, Theorem 3.31 becomes

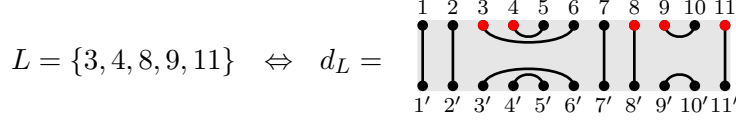
$$\mathbf{M}_{\mathbf{TL}_k}^{(k-2\ell)} \cong \mathbf{TL}_k^{(k-2\ell)} \quad \text{and} \quad \mathbf{M}_{\mathbf{TL}_k} \cong \bigoplus_{\ell=0}^{\lfloor k/2 \rfloor} \mathbf{M}_{\mathbf{TL}_k}^{(k-2\ell)} \cong \bigoplus_{(k-2\ell) \in \Lambda_{\mathbf{TL}_k}} \mathbf{TL}_k^{(k-2\ell)}. \quad (4.24)$$

If we let $\mathbf{tl}_k = |\mathbf{I}_{\mathbf{TL}_k}| = \sum_{\ell=0}^{\lfloor k/2 \rfloor} |\mathbf{I}_{\mathbf{TL}_k}^{k-2\ell}| = \dim \mathbf{M}_{\mathbf{TL}_k}$ denote the total number of symmetric diagrams in $\mathbf{TL}_k(x)$, then \mathbf{tl}_k is the sum of the degrees of the irreducible $\mathbf{TL}_k(x)$ -modules. There is a bijection between the symmetric Temperley-Lieb diagrams $\mathbf{I}_{\mathbf{TL}_k}$ and subsets of $\{1, 2, \dots, k\}$ of size $\lfloor k/2 \rfloor$ as follows:

- For $d \in \mathbf{I}_{\mathbf{TL}_k}$ with $0 \leq c \leq \lfloor k/2 \rfloor$ contractions, let L be the set of left endpoints of the contractions in the top row of d . Then add to L the $\lfloor \frac{k}{2} \rfloor - c$ rightmost fixed points of d . This constructs a subset $L \subseteq \{1, 2, \dots, k\}$ with $|L| = \lfloor k/2 \rfloor$ that we call the left endpoint set of d .

- Inversely, let $L \subseteq \{1, 2, \dots, k\}$ with $|L| = \lfloor k/2 \rfloor$, and start with an empty diagram d_L . Working from the largest to smallest elements of L , connect vertex $i \in L$ in the top row of d to the smallest empty vertex $j \in \bar{L}$ in the top row of d such that $j > i$. If there are no such vertices, connect i to i' . Mirror all contractions in the bottom row, and for any remaining empty vertices $j \in \bar{L}$ connect j to j' . In this way, we are able to construct a diagram d_L whose left endpoint set is L .

For example, in $\text{TL}_{11}(x)$ the set $L = \{3, 4, 8, 9, 11\}$ corresponds to the diagram d_L below:



The diagram d_L is constructed as follows. Starting with vertex 11; there is no vertex to its right so we connect it to $11'$. Vertex 9 is connected to the smallest vertex 10 to its right that is not in L . There is no available vertex to the right of 8 so we connect it to $8'$. Then 4 is connected to 5, the smallest vertex to its right not in L , and finally 3 is connected to 6 for the same reason.

It follows that $\text{tl}_k = \binom{k}{\lfloor k/2 \rfloor}$ (the k th central binomial coefficient), which is [OEIS] Sequence A000984, and whose first few values are

k	0	1	2	3	4	5	6	7	8	9	10
$\text{tl}_k = \dim \mathbf{M}_{\text{TL}_k}$	1	1	2	3	6	10	20	35	70	126	252

(4.25)

This sequence has exponential generating function

$$I_0(2x) + I_1(2x) = \sum_{k=0}^{\infty} \text{tl}_k \frac{x^k}{k!}, \quad (4.26)$$

where $I_n(z)$ is the modified Bessel function of the first kind (see for example [GKP, (5.78)]).

Remark 4.27. The irreducible modules $\text{TL}_k^{(k-2\ell)}$ are constructed by Westbury [We] on “cup diagrams” (or 1-factors). These cup diagrams correspond exactly to the upper half of a symmetric diagram (since the diagrams are symmetric, only half is needed), and Westbury’s action of $\text{TL}_k(x)$ on these diagrams is exactly the same as our conjugation action on symmetric diagrams.

4.6 The Motzkin algebra $\mathbf{M}_k(x)$

The Motzkin algebra $\mathbf{M}_k(x)$ is the subalgebra of $\mathbf{P}_k(x)$ spanned by Motzkin diagrams (planar rook-Brauer diagrams). Motzkin diagrams correspond to partial planar matchings of $\{1, \dots, k, 1', \dots, k'\}$, and so the dimension of $\mathbf{M}_k(x)$ is the Motzkin number M_{2k} (see [BH]). The Motzkin algebra $\mathbf{M}_k(x)$ is semisimple for $x \in \mathbb{C}$ chosen such that x is not the root of the polynomial $U_k((x-1)/2)$ where U_k is the Chebyshev polynomial of the second kind (see [BH]). If, we let $x \in \mathbb{C}$ be chosen so that $\mathbf{M}_k(x)$ is semisimple, then the irreducible modules are indexed by the following set of integers

$$\Lambda_{\mathbf{M}_k} = \{0, 1, \dots, k\}. \quad (4.28)$$

Let $\mathbf{M}_k^{(r)}$ denote the irreducible module labeled by $r \in \Lambda_{\mathbf{M}_k}$, and let $\chi_{\mathbf{M}_k}^{(r)}$ denote its character.

Symmetric Motzkin diagrams in $M_k(x)$ consist of f fixed points, c contractions and $k - f - 2c$ pairs of empty vertices. For example, the symmetric Motzkin diagram,



$$\in M_{14}(x)$$

has $c = 4$ contractions in positions $\{1, 2\}, \{4, 9\}, \{6, 8\}, \{12, 13\}$, $f = 3$ fixed points in positions 3, 10, 14, vertical pairs of empty vertices in positions 5, 7, 11, and rank $r = 3$. Observe that $r = f$ and that every possible rank $0 \leq r \leq k$ is possible. The number of symmetric diagrams of this type is

$$\dim M_{M_k}^r = |I_{M_k}^r| = \sum_{c=0}^{\lfloor (k-r)/2 \rfloor} \binom{k}{r+2c} \left\{ \begin{matrix} r+2c \\ c \end{matrix} \right\}, \quad (4.29)$$

where $\left\{ \begin{matrix} r+2c \\ c \end{matrix} \right\}$ is defined in (4.23). This formula is derived in [BH, (3.21)].

For the Motzkin monoid algebra, Theorem 3.31 becomes

$$M_{M_k}^r \cong M_k^{(r)} \quad \text{and} \quad M_{M_k} \cong \bigoplus_{r=0}^k M_{M_k}^r \cong \bigoplus_{r \in \Lambda_{M_k}} M_k^{(r)}. \quad (4.30)$$

If we let $m_k = |I_{M_k}| = \sum_{r=0}^k |I_{M_k}^r| = \dim M_{M_k}$ denote the total number of symmetric diagrams in $M_k(x)$, then m_k is the sum of the degrees of the irreducible $M_k(x)$ -modules. The first few values of m_k are

k	0	1	2	3	4	5	6	7	8	9	10
$m_k = \dim M_{M_k}$	1	2	5	13	35	96	267	750	2123	6046	17303

$$(4.31)$$

The sequence m_k is [OEIS] Sequence A005773 and it is related to the number of symmetric diagrams tl_k in the Temperley-Lieb algebra (4.25) by the binomial transform $m_k = \sum_{i=0}^k \binom{k}{i} tl_i$. Thus (see [GKP, (7.75)]) m_k has exponential generating function

$$e^x(I_0(2x) + I_1(2x)) = \sum_{k=0}^{\infty} m_k \frac{x^k}{k!}. \quad (4.32)$$

Remark 4.33. The irreducible modules $M_k^{(r)}$ are constructed in [BH] on 1-factors. These 1-factors correspond exactly to the upper half of a symmetric Motzkin diagram, and the action of $M_k(x)$ on these diagrams is exactly the same as our conjugation action on symmetric diagrams. Indeed, it was knowledge of this conjugation action that allowed [BH] to produce the action of $M_k(x)$ on 1-factors.

4.7 The planar rook monoid algebra PR_k

The planar rook monoid algebra PR_k is the subalgebra of $P_k(x)$ spanned by planar rook-monoid diagrams with parameter set to $x = 1$. It has dimension $\binom{2k}{k}$, and is semisimple with irreducible modules labeled by

$$\Lambda_{PR_k} = \{0, 1, \dots, k\}. \quad (4.34)$$

Let $PR_k^{(r)}$ denote the irreducible module labeled by $r \in \Lambda_{PR_k}$, and let $\chi_{PR_k}^{(r)}$ denote its character.

Symmetric planar rook monoid diagrams in PR_k consist of f fixed points and $k - f$ vertical pairs of empty vertices. For example, the symmetric planar rook monoid diagram,



$$\in PR_{14}$$

has $f = 7$ fixed points in positions 2, 3, 5, 8, 10, 11, 14, and rank $r = 7$. We associated this diagram with its set of fixed points $S = \{2, 3, 5, 8, 10, 11, 14\}$, and thus symmetric diagrams correspond exactly to subsets $S \subseteq \{1, 2, \dots, k\}$. Thus, the number of symmetric diagrams is

$$\dim M_{\text{PR}_k}^{r,f} = \dim M_{\text{PR}_k}^{r,r} = |I_{\text{PR}_k}^r| = \binom{k}{r}. \quad (4.35)$$

For the planar rook monoid algebra, Theorem 3.31 becomes

$$M_{\text{PR}_k}^r \cong \text{PR}_k^{(r)} \quad \text{and} \quad M_{\text{PR}_k} \cong \bigoplus_{r=0}^k M_{\text{PR}_k}^r \cong \bigoplus_{r \in \Lambda_{\text{PR}_k}} \text{PR}_k^{(r)}. \quad (4.36)$$

If we let $\text{pr}_k = |I_{\text{PR}_k}| = \sum_{r=0}^k |I_{\text{PR}_k}^r| = \dim M_{\text{PR}_k}$ denote the total number of symmetric diagrams in PR_k , then pr_k is the number of subsets of $\{1, 2, \dots, k\}$, so

$$\text{pr}_k = \dim M_{\text{PR}_k} = 2^k. \quad (4.37)$$

Remark 4.38. *The irreducible modules $\text{PR}_k^{(r)}$ are constructed in [FHH] on a basis of r -subsets of $\{1, 2, \dots, k\}$. These r -subsets correspond to symmetric rook monoid diagrams, and the action of PR_k on subsets is exactly the same as our conjugation action on symmetric diagrams. Indeed, it was knowledge of this conjugation action that [FHH] to produce the action of PR_k on subsets.*

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